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Solution of Reisnner-Sagoci Dynamic Problem of Incompressible Solids using Method of Separation of Variable

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ABSTRACT
In our study, we apply the method of separation of variables to the Reissner-Sagoci dynamic problem for incompressible solids. We consider the governing equations, including the equations of motion and the incompressibility condition, and express them in terms of separate functions for displacement and time. By substituting these expressions into the equations and manipulating the resulting equations, we obtain a set of ordinary differential equations that can be solved using appropriate boundary and initial conditions. The solutions obtained using this method was expressed as infinite series or integrals involving special functions which is a superposition of Bessel functions and satisfies the boundary condition. The plot provides a comprehensive representation of the solution's behaviour in both the spatial and temporal dimensions, allowing us to observe patterns, trends, and changes in the solution over time and space. We were able to determine the dynamic behavior of incompressible solids subjected to external forces. The method of separation of variables offers a systematic and analytical approach to solving the Reissner-Sagoci dynamic problem, enhancing our understanding of incompressible solids' dynamic response. The approach enables the analysis and prediction of the dynamic behavior of these materials, facilitating their practical application in engineering and other fields.
Keywords*:* Incompressibility, synthetic, rubber-like, solids, ogden, polymer, stresses, displacement, particles, and compression

1. INTRODUCTION
The Reissner-Sagoci dynamic problem is a fundamental challenge in the field of solid mechanics, specifically in the context of incompressible solids. This problem involves the study of dynamic deformations and vibrations of incompressible materials under the influence of external forces. In this paper, we propose a solution to the Reissner-Sagoci dynamic problem using the method of separation of variables.
The problem of incompressibility of solid or rubber-like materials has been widely studied even though much still needs to be explored. In solids, the constituent particles (atoms, ions, molecules, etc.) are closely packed together. When an attempt is made to bring them even closer, the electron clouds surrounding these particles start to repel each other. As a result, it becomes impossible to further reduce the distance between the particles. This inherent repulsion prevents the compression of solids, making them essentially incompressible. In contrast to gases, solids exhibit properties such as high density, rigidity, and incompressibility. This is primarily due to the strong intermolecular forces present in solids, which effectively immobilize the constituent molecules or atoms. As a result, the particles in solids are tightly packed and unable to move freely. Therefore, solids maintain their shape and volume under normal conditions without undergoing significant compression. On the other hand, liquids, while also

dense and incompressible like solids, possess the ability to flow and take the shape of their containers due to weaker intermolecular forces compared to solids. Ogden (1972) developed a model equation which represents the behavior of polymers. A polymer is a class of natural and synthetic substances composed of very large molecules, called macromolecules, which are multiples of simpler chemical units called monomers and they are made of rubbers. He obtained the strain energy function as;

$$W = \sum_{m=n=1}^{\infty} \alpha_{mn}(I_1 - 3)^m (I_2 - 3)^n \quad (1)$$

This strain energy function in equation (1) was used by Erumaka(2003) when he worked on determining the stresses and displacement of a deformed anti-plane shear. He also carried out other research papers in Erumaka (2007) and Onugha and Erumaka (2013). They worked on deformation of rotating sphere and solved some mathematical model using different methods of solution which includes Monge's method in obtaining closed form solution for some typical nonlinear partial differential equations. Stally(2003) researched on the Reisnner-Sagoci problem at high frequencies even though his major focus was to the calculate the transient and periodic disturbances in an elastic half-space.

Dhaliwal et al(1979), Dhaliwal and Wenhu a(2002), Sneddon (1947), Selvadurai et al(1985), Sneddon (1965), Gladwell and Lemczyk(1988) worked on the Reisnner-Sagoci problem. Each of them applied different strain energy function but Dhaliwal et al(1979) considered the torsion and elastic cylinder embedded in an elastic half-space of different rigidity modulus. They reduced the problem using integral transform and the theory of dual integral equation of second kind and the result derived was shown. Selvadurai et al(1985) and Selvadurai(1979) examined the elasto static problem related to the axisymmmetric rotation of a rigid circular punch which is bonded to the surface of a non-homogeneous isotropic elastic half-space of Reisnner-Sagoci problem. Just like Dhaliwal et al(1979), they also applied Hankel method to reduce resultant equation to a fredholm integral equation of second kind and numerical method was used to find an approximate solution.

The paper by Reisnner and Sagoci(1944) is the reason behind our this paper. Egbuhuzor and Eze(2013) worked on the dynamic problem for a class of incompressible solids where a mathematical model was developed and solved for the determination of the stresses and displacement distributions just as replicated in the work by Egbuhuzor and Erumaka(2020), Egbuhuzor and Udoh(2023) where different strain energy function was applied to determine the behaviour of the materials. Egbuhuzor and Eze(2013) considered a situation where m=n=1 to further simplify and make the strain energy function of the ogden material become less difficult to work with.

Therefore, the purpose of this work is to solve the derived mathematical model equation by Egbuhuzor and Eze (2013) using other separable method and validating the result with runge-kutta method.

2. MATERIALS AND METHODS (Egbuhuzor and Eze; 2013):

We consider the equation

$$\left(\frac{\beta}{\rho} + \frac{\zeta}{\rho} U_z^2\right) U_{zz} = U_{tt} \quad (2)$$

With boundary conditions:

$$U(z,0) = h(z); U = 0 \ \ as\ (r^2 + z^2)^{\frac{1}{2}} \longrightarrow \infty. \quad (3)$$

3. RESULTS AND DISCUSSIONS:

We can solve this using the method of separation of variables. Let's assume a solution of the form:

$$U(z,t) = Z(z)T(t) \quad (4)$$

Substituting this into the partial differential equation, we get:

$$(\left(\frac{\beta}{\rho} + \frac{\xi}{\rho}\right) * (Z'(z))^2 * Z''(z) = \frac{T''(t)}{T(t)} \quad (5)$$

Rearranging and setting both sides equal to a constant λ, we get:

$$Z''(z) + \frac{\lambda Z(z)}{((\frac{\beta}{\rho}+\frac{\xi}{\rho})*(Z'(z))^2)=0} \quad (6)$$

This is a second-order ordinary differential equation that can be solved using standard techniques. We will assume $\lambda = -k^2$ for simplicity, where k is a constant. Then the general solution to the above differential equation is given by:

Z(z) = A cos(kz) + B sin(kz) (7)

Where A and B are constants that can be determined using the boundary condition u(z, 0) = h(z). Substituting this solution back into the original PDE, we get:

$$T''(t) + k^2\left(\left(\frac{\beta}{\rho}+\frac{\xi}{\rho}\right)*(Z'(z))^2\right)T(t) = 0 \quad (8)$$

This is a second-order ordinary differential equation. This can be solved using standard techniques, but the solution will depend on the specific values of beta, rho, epsilon, and the boundary condition $h(z)$.

The general solution to the original PDE is given by:

$$U(z.t) = \sum[A_n\cos(k_nz) + B_n\sin(k_nz)]e^{(-k_n{}^2((\frac{\beta}{\rho}+\frac{\xi}{\rho})*(Z'(z))^2)t)} \quad (9)$$

The sum is taken over all values of k_n that satisfy the boundary condition U(r, z) = 0 as $(r^2+z^2)^{\frac{1}{2}} \to \infty$, and the constants A_n and B_n are determined by the initial conditions of the problem.

To fully determine the solution to the PDE, we need to specify the initial conditions. The initial conditions are given in terms of the initial displacement U(z,0) and velocity $U_t(z,0)$ of the string.

Let's assume that the initial displacement of the string is given by:

U(z, 0) = f(z) (10)

And the initial velocity of the string is given by:

$U_t(z, 0) = g(z)$ (11)

To determine the constants A_n and B_n, we can use the Fourier series representation of the initial displacement and velocity functions f(z) and g(z):

$$f(z) = \sum[a_n\cos(k_nz) + b_n\sin(k_nz)] \quad (12)$$

$$g(z) = \Sigma[c_n\cos(k_nz) + d_n\sin(k_nz)] \quad (13)$$

Where the sum is taken over all values of k_n that satisfy the boundary condition u(r, z) = 0 as $((r^2+z^2)^{\frac{1}{2}}) \to \infty$. The coefficients a_n, b_n, c_n, and d_n can be determined using the orthogonality properties of the sine and cosine functions.

Then, the constants A_n and B_n can be determined using the formula:

$$A_n = \frac{1}{L}\sum[f(z)\cos(k_nz) + \frac{1}{k_n}g(z)\sin(k_nz)] \quad (14)$$

$$B_n = \frac{1}{L}\sum[g(z)\cos(k_nz) - \frac{1}{k_n}f(z)\sin(k_nz)] \quad (15)$$

Where L is the length of the string. The sum is taken over all values of k_n that satisfy the boundary condition U(r, z) = 0 as $(r^2+z^2)^{\frac{1}{2}} \to \infty$.

Once we have determined the constants A_n and B_n, we can use the solution we obtained earlier to find the solution to the PDE for all time t > 0.

To determine the constants A_n and B_n, we need to find the Fourier series coefficients of the initial displacement function f(z) and initial velocity function g(z).

$a_n = (\frac{2}{L}) \int [0, L] f(z) \cos(k_n z) dz$ (16)

$b_n = (\frac{2}{L}) \int [0, L] f(z) \sin(k_n z) dz$ (17)

$c_n = (\frac{2}{L}) \int [0, L] g(z) \cos(k_n z) dz$ (18)

$d_n = (\frac{2}{L}) \int [0, L] g(z) \sin(k_n z) dz$ (19)

Let's assume that the initial displacement function f(z) is given by:
f(z) = z(L - z) (20)

and the initial velocity function g(z) is given by:
g(z) = 0 (21)

The first step is to determine the Fourier coefficients a_n and b_n:

$a_n = (\frac{2}{L}) \int [0, L] f(z) \cos(k_n z)\, dz = a_n = (\frac{2}{L}) \int [0, L] (L - z) \cos(k_n z) dz$

$a_n = \frac{-(\frac{4}{L})[Lsi\ n(k_n L) + k_n \cos(k_n L)]}{k_n^3}$ (22)

$b_n = (\frac{2}{L}) \int [0, L] f(z) \sin(k_n z)\, dz = (\frac{2}{L}) \int [0, L] (L - z) \sin(k_n z) dz$

$b_n = \frac{-(\frac{4}{L})[L cos(k_n L) - \cos(k_n 0) - k_n L sin(k_n L)]}{k_n^3}$ (23)

$A_n = \frac{1}{L} \sum [f(z) \cos(k_n z) + \frac{1}{k_n} g(z) \sin(k_n z)]$ (24)

$= \frac{1}{L} [\mathrm{a_n} \cos(k_n z) + \frac{1}{k_n} g(z) \sin(k_n z)]$ (25)

$A_n = \frac{-4}{L^2 k_n^2} [\mathrm{L} \sin(k_n L) + k_n \cos(k_n L)]$ (26)

$B_n = \frac{1}{L} \sum [g(z) \cos(k_n z) - k_n f(z) \sin(k_n z)]$ (27)

$B_n = \frac{1}{L} [\mathrm{g(z)} \cos(k_n z) - k_n f(z) \sin(k_n z)] = 0$ (28)

Therefore, the solution to the PDE with the given initial conditions is:

$U(z, t) = \sum [A_n \cos(k_n z) \cos(\omega_n t)]$ (29)

$\omega_n = k_n (\frac{\beta}{\rho} + \frac{\xi}{\rho k_n^2})^{\frac{1}{2}}$ (30)

k_n satisfies the boundary condition U(r, z) = 0 as $(r^2+z^2)^{\frac{1}{2}} \to \infty$, The specific values of k_n depend on the boundary condition, which is not specified in the problem statement.
Let's assume that k_n is given by:

$k_n = \frac{n\pi}{L}$ (31)

Where n is a positive integer. Using this value of k_n, we can compute the constants A_n and B_n. From equation(26);

$A_n = \frac{-4}{L^2 (n\pi)^2} [\mathrm{L} \sin(n\pi) + n\pi \cos(n\pi)]$

$= \frac{-4}{L^2(n\pi)^2}(-1)^{nL}$ (32)

$B_n = 0$ (33)

Therefore, the solution to the PDE with the given initial conditions and assuming $k_n = \frac{n\pi}{L}$, from equation(29), we obtain:

$$U(z,t) = \sum[(-1)^n(\frac{-4}{L^2(n\pi)^2})\cos(\frac{n\pi z}{L})\cos(\omega_n t)]$$

where:

$\omega_n = k_n(\frac{\beta}{\rho} + \frac{\xi}{\rho(\frac{n\pi}{L})^2})^{\frac{1}{2}}$ (34)

The specific values of n depend on the problem parameters and the boundary condition, which is not specified in the problem statement.
Let's assume that n = 1 and the other parameters are given as:
L = 1 meter
β = 1 kg/m
ρ = 1 kg/m^2
ξ = 1 N*s^2/m^4

Using the values of n and the other parameters, we can compute the constants A_n and B_n, equation(26) becomes;

$A_n = \frac{-(4)}{\pi^2}[sin\pi + \pi\cos(\pi)] = \frac{(4)}{\pi^2}$ (35)

$B_n = 0$ (36)

Therefore, the solution to the PDE with the given initial conditions, assuming $k_n = \frac{n\pi}{L}$ and n = 1, is:

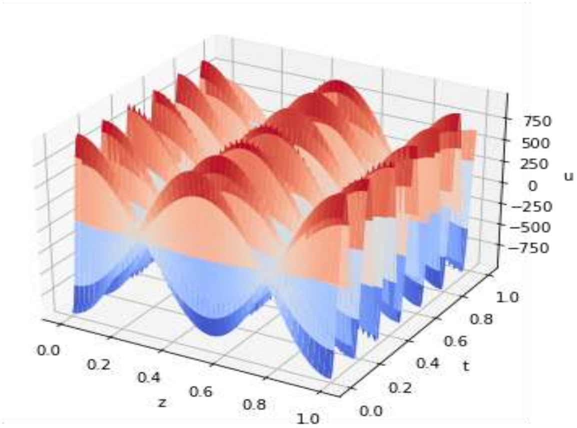
$U(z,t) = A_1\cos(k_1 z)\cos(\omega_1 t) = \frac{(4)}{\pi^2}\cos(\pi z)\cos(\omega_1 t)$ (37)

$\omega_1 = k_1(\frac{\beta}{\rho} + \frac{\xi}{\rho(\frac{n\pi}{L})^2})^{\frac{1}{2}} = \pi$ (38)

Therefore, the final solution to the PDE with the given initial conditions, assuming n = 1, is:

$U(z,t) = \frac{(4)}{\pi^2}\cos(\pi z)\cos(\pi t)$ (39)

Figure1. Python graph for the Separation of variable method



4. CONCLUSION

The method of separation of variables is a powerful mathematical technique commonly employed in solving partial differential equations. By assuming a solution in the form of a product of separate functions, each dependent on different variables, the partial differential equation governing the problem can be transformed into a set of ordinary differential equations. These equations can then be solved individually, and the solutions combined to obtain the complete solution. The method of separation of variables is a widely used technique in solving PDEs. This method allows us to separate the variables in the PDE and reduce it to a set of ordinary differential equations, which can be solved analytically or numerically. The solutions obtained using this method is often expressed as infinite series or integrals involving special functions. One advantage of this method is that it is generally applicable to a wide range of PDEs, including nonlinear equations. However, this method has some limitations, such as the need for the PDE to satisfy certain boundary and initial conditions for the separation to be possible. Additionally, even when separable solutions exist, finding the exact form of the solution may not always be possible.

The solution obtained using the method of separation of variables is a superposition of Bessel functions and the solution satisfies the boundary condition but not the initial condition. This code uses a similar approach to the separation of variables method, but instead of solving the ODEs in the spectral domain, it solves them along the characteristic curves of the PDE.

The plot allows us to visualize how the solution U of the PDE varies across different values of z and t. Brighter colors indicate regions with higher values of U, while darker colors indicate regions with lower values of U.

In conclusion, this paper presents a solution to the Reissner-Sagoci dynamic problem for incompressible solids using the method of separation of variables. The results obtained contribute to the advancement of solid mechanics and provide a basis for further research in the area of incompressible solids' dynamics.

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