International Journal of Innovative Mathematics, Statistics & Energy Policies
7(1):1-18, Jan.-Mar., 2019

 www.seahipaj.org ISSN: 2467-852X

The Efficiency of Qr Factorization Method for Eigenvalues Approximations

OJO-OROBOSA V.O.

Department Of Mathematics and Statistics
Delta State Polytechnic, Ozoro, Nigeria
Email: orobosavera@gmail.com.
08168851183

ABSTRACT
We consider the problem of computing all the eigenvalues of a real symmetric matrix $A = \{ajk\}$, by discussing a technique widely used in practice which is the QR factorization. This method is very efficient and commonly implemented in various algorithms that uses matrix factorization. In this regard we shall first apply the householder's method to reduce the given matrix to a tridiagonal matrix, thereafter the matrix is factorized in the QR form where Q is Orthogonal and R upper triangular. We demonstrate the efficiency of this approach by constructing a 3 QR-steps to find approximation to the Eigen values for general real or complex matrices. This is achieved by considering real symmetric tridiagonal matrix $P_0 = P$ via Householder's method;

Keywords: QR Factorization, Tridiagonal Matrix, Real Symmetric Matrix, Orthogonal, Householder's Method, Complexity and Cycle.

INTRODUCTION
The QR-factorization method is an improved version of Rutishauser's approach (avoiding breakdown if certain sub matrices become singular [1] which is based on the factorization QR, where R is upper triangular as in LR-factorization but Q is orthogonal (instead of lower triangular). This method is a very stable numerical approach for finding approximations of all eigenvalues of a given matrix, as opposed to methods such as the Rayeleigh Quotient, the power method, the inverse power method which can only find a single eigenvalue. This attribute make the QR-factorization techniques unique and very powerful [2].
It will be essential for us to discuss briefly eigenvalues. Eigen values problems are among the most prominent problem in connection with matrices. For a square matrix A, consider the equation of the form

$$A = \lambda x \qquad (1)$$

Where λ is an unknown scalar and x is an unkown vector. In solving (1) above, if $x = 0$ then the solution is trival, it has no meaning to us but if $x \neq 0$ then there exist a solution only for certain values of λ, then these values are called eigenvalues (or characteristic values) of A
The evaluation of eigenvalues is very paramount to engineers, physicists and computer Analysts. The major goal in this approach is to be able to represent any $m \times n$ matrix in the form

$$A = QR \qquad (2)$$

Where Q is an orthogonal[1] $m \times n$ matrix such that its columns constitute an orthonomal[2] basis for Col A, and R is an $n \times n$ invertible upper triangular matrix with positive entries on its diagonal [3].

Now we shall discuss the QR technique for a real symmetric matrix (for extension to general real or complex matrices). Let's begin from a real symmetric tridiagonal matrix $P_0 = P$ (as derived from a real symmetric matrix A by Householder's method) [4] we then compute step wise $P_1, P_2, \dots$ according to this rule:

First step factor

$P_0 = Q_0 R_0$

Then compute

$P_1 = R_0 Q_0$

Second step factor

$P_1 = Q_1 R_1$

Then compute

$P_2 = R_1 Q_1$

General step factor

$$P_s = Q_s R_s \qquad (3)$$

Where Q_s is orthogonal and R_s is upper triangular

Then compute

$$P_{s+1} = R_s Q_s \qquad (4)$$

The method of factorization (3) is explained as;

P_{s+1} similar to P convergence to a diagonal matrix. From (3) we have $R_s = Q_s^{-1} P_s$ substituting into (4), we have

$$P_{s+1} = R_s Q_s = Q_s^{-1} P_s Q_s \qquad (5)$$

This imply that P_{s+1} is similar to P_0. Hence P_{s+1} is similar to $P_0 = P$ for all s. this also means that P_{s+1} has the same Eigen values as P.

Also P_{s+1} is symmetric. This follows by induction, $P_0 = P$ is symmetric. Assuming P_s to be symmetric and using

$Q_s^{-1} = Q^T$ since Q_s is orthogonal)

We obtained from (5) above

$$P_{s+1}^T = (Q_s^T P_s Q_s)^T = Q_s^T P_s^T Q_s = Q_s^T P_s Q_s = P_{s+1} \qquad (6)$$

As obtained.

If the eigenvalues of P are different in absolute value say

$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Then

$$\lim_{s \to \infty} P_s = D \qquad (7)$$

Where D is diagonal with main diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$. [1]

This algorithm would be cumbersome when performing manually.

The QR factorization can accurately be perform using various computer based implementations which are applicable to every numeric algorithm that computes eigenvalues. [2]

We organized this article as follows; section 2 contains preliminaries related to our main result. In section 3 we shall compute column k and depict illustration by way of example and also demonstrate the construction of a 3 QR- step factorization for a given matrix. The algorithm for QR factorization and Householder's algorithm is also provided in section 3. In section 4, numerical experiment and some examples involving complexity will be given. While summary and conclusion is given section 5.

2.0 Preliminaries;

This section contains important concepts which will be used in the forthcoming sections

2.1 Householder's method:

The Householder's method reduces the given matrix to a tridiagonal matrix. That is a matrix having all its nonzero entries on the main diagonal and in the position immediately adjacent to the main diagonal. The tridiagonalization process reduces a real symmetric $n \times n$ matrix $A = \{a_{jk}\}$ by $n-2$ successive similarity transformations to tridiagonal form. Considering the matrices $H_1, H_2, \dots, H_{n-2}$ are orthogonal and symmetric, hence $H_1^T = H_1$ and similarly for the others [6]. The $n-2$ similarity transformations that successively produced from the given $A_0 = A = \{a_{jk}\}$, the matrices $A_1 = \{a_{jk}^1\}, A_2 = \{a_{jk}^2\}$, etc now takes the form

$A_1 = E_1\, A_0\, E_1$

$A_2 = E_2\, A_1\, E_2$

-- -- -- -- --

$$P = A_{n-2} = E_{n-2} A_{n-3} E_{n-2} \qquad (8)$$

It is this transformations that creates the necessary zeros in the first step in row 1 and column 1, also in the second step in row 2 and column 2, etc.

$$\Rightarrow E_r = 1 - 2v_r v_r^T \quad (r = 1, \dots\dots, n-2) \qquad (9)$$

By way of application and manual computation, there are three major steps [1].

Step 1. V_1 has the components

$$V_{11} = 0$$

(a) $V_{21} = \sqrt{\frac{1}{2}\left(1 + \frac{|a_{21}|}{S_1}\right)}$

(b) $V_{j1} = \frac{a_{j1}\, sgn\, a_{21}}{2V_{21}\, S_1}$

(c) Where $S_1 = \sqrt{a_{21}^2 + a_{31}^2 + \cdots + a_{n1}^2}$ (10)

Where $S_1 > 0$, and $sgn\, a_{21} = +1, if\ a_{21} \geq 0\ and\ sgn\, a_{21} = -1, if\ a_{21} < 0$. with this we compute E_1 by (9) and then A_1 by (8).

In the second step, we compute V_2 by (10) with all subscript increased by 1 and the a_{jk} replaced by a_{jk}^1, the entries of A_1 just computed

$$V_{12} = V_{22} = 0$$

$$V_{31} = \sqrt{\frac{1}{2}\left(1 + \frac{|a_{32}|}{S_2}\right)}$$

$$V_{j1} = \frac{a_{j2}^{(1)} sgn\, a_{32^{(1)}}}{2V_{32}\, S_2}, j = 4, 5, \dots n$$

$$\text{Where } S_1 = \sqrt{a_{32}^{(1)\,2} + a_{42}^{(1)\,2} + \cdots + a_{n2}^{(1)\,2}} \qquad (11)$$

With this we compute E_2 by (9) and then A_2 by (8).

Third step, we compute V_3 by (11) with all subscripts increased by 1 and the $a_{jk}^{(1)}$ replaced by the entries $a_{jk}^{(2)}$ of A_2, and so on

Considering a real symmetric matrix

$$A = \begin{bmatrix} 6 & 4 & 1 & 1 \\ 4 & 6 & 1 & 1 \\ 1 & 1 & 5 & 2 \\ 1 & 1 & 2 & 5 \end{bmatrix}$$

Applying the above steps, we can easily tridiagonalized the matrix as;

$S_1^2 = 4^2 + 1^2 + 1^2 = 18,$ from (10c), since $a_{21} = 4 > 0,$ we have that $sgn\, a_{21} = +1$ in (10b) and also from (10) by straight forward computations, we get

$$V_1 = \begin{bmatrix} 0 \\ V_{21} \\ V_{31} \\ V_{41} \end{bmatrix} = \begin{bmatrix} 0. \\ 0.98559856 \\ 0.11957316 \\ 0.11957316 \end{bmatrix}$$

From this and (9)

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.94280904 & -0.23570227 & -0.23570227 \\ 0 & -0.23570227 & 0.97140452 & -0.02859748 \\ 0 & -0.23570227 & -0.02859748 & 0.97140452 \end{bmatrix}$$

From (8) we get

$$A_1 = E_1 A_0 E_1 = \begin{bmatrix} 6 & -\sqrt{18} & 0 & 0 \\ -\sqrt{18} & 7 & -1 & -1 \\ 0 & -1 & \frac{9}{2} & \frac{3}{2} \\ 0 & -1 & \frac{3}{2} & \frac{9}{2} \end{bmatrix}$$

From (11) we get $S_2^2 = 2$ and

$$V_2 = \begin{bmatrix} 0 \\ 0 \\ V_{32} \\ V_{42} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.92387953 \\ 0.38268343 \end{bmatrix}$$

Applying this and (9)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Also from (8) we have

$$P = A_2 = E_2 A_1 E_2 = \begin{bmatrix} 6 & -\sqrt{18} & 0 & 0 \\ -\sqrt{18} & 7 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 6 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The matrix P is tridiagonal. The given matrix has order $n = 4$, so we needed $n - 2$ steps to accomplish our goal. However it is obvious that there are more zeros than we can expect in general [3].

An efficient sequential algorithm for the above computation is for $i = 1; n - 2$

$$v = House\big(A(i+1{:}n; i)\big)$$

$$v = \sqrt{\frac{2}{v^T v}}v$$

$$E = A(i+1{:}n, i+1{:}n)v$$

$$w = E - \frac{1}{2}(E^T v)v$$

$$A(i+1{:}n, i+1{:}n) = A(i+1; n, i+1{:}n) - vw^T - wv^T$$

End.

The resulting algorithm requires $4n^3/3 + 0(n^2) flops$ [4]

2.2 QR Factorization

If $A \in R^{m \times n}$ and has linearly independent column then it can be factors as

$$A = \{q_1, q_2, \dots, q_n\} \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ 0 & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{mn} \end{bmatrix}$$

Where $q_1, \dots q_n$ are orthonormal m-vectors

$||q_i|| = 1, q_i^T q_j = 0$ if $i \neq j$

The diagonal elements R_{ii} are non-zero. However if $R_{ii} < 0$, we can switch the signs of $R_{ii}, \dots, R_{in}$ but if $R_{ii} > 0$, it makes Q and R unique. The usefulness of this method is more obvious when dealing with large problem sizes which would completely impossible with other techniques [4].

To obtain the QR factorization say for $P = P_0 = \{p_{jk}\} = Q_0 R_0$, the tridiagonal matrix P follows from section 1 has $n-1$ generally nonzero entries below the main diagonal. These are $p_{21}, p_{32}, \ldots p_n, n-1$.

Multiplying P from the left by a matrix C_2 such that $C_2 P = \{p_{jk}^{(2)}\}$ has $p_{21}^{(2)} = 0$. Again if we multiply this by a matrix C_3 such that $C_3 C_2 P = \left\{p_{jk}^{(3)}\right\} has\ p_{32}^{(3)} = 0$ etc.

After $n-1$ of such multiplication we have an upper triangular matrix R_0

i.e $C_n C_{n-1} \ldots C_3 C_2 P_0 = R_0$ $\quad (12)$

These C_j are very simple and has 2×2 submatrix

$$\begin{bmatrix} cos\theta_j & sin\theta_j \\ -sin\theta_j & cos\theta_j \end{bmatrix} (\theta_j\ suitable)$$

This sub matrix is a matrix of plane rotation through the angle θ_j. These C_j are orthogonal and their product in (12) is orthogonal and so is the inverse of this product and this is called Q_0. Also from (12)

$$P_0 = Q_0 R_0 \quad (13)$$

Where with $C_j^{-1} = C_j^T$

$$Q_0 = (C_n C_{n-1}, \ldots, C_3 C_2)^{-1} = C_2^T C_3^T, \ldots, C_{n-1}^T C_n^T \quad (14)$$

(14) is the QR-factorization of P_0

Also by (4) with $s = 0$, we have

$$P_1 = R_0 Q_0 = R_0 C_2^T C_3^T \ldots, C_{n-1}^T C_n^T \quad (15)$$

To get P_1 from (15), we compute $R_0 C_2^T$, then$(R_0 C_2^T) C_3^T$, similarly we do same to get $P_2, P_3 \ldots$

$$CP = \begin{bmatrix} C_2 & S_2 & 0 & \dots \\ -S_2 & C_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \dots \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \dots \\ p_{21} & p_{22} & \dots \\ \vdots & \vdots & \dots \end{bmatrix}$$

Now $p_{21}^{(2)}$ is obtained by multiplying the second row of C_2 by the first column of P,

$$p_{21}^{(2)} = -S_2 p_{11} + C_2 p_{21} = 0$$

Hence $\tan\theta_2 = \frac{S_2}{C_2} = \frac{p_{21}}{p_{11}}$ and

$$\cos\theta_2 = \frac{1}{\sqrt{1 + \left(\frac{p_{21}}{p_{11}}\right)^2}},$$

$$(16)$$

$$\sin\theta_2 = \frac{\frac{p_{21}}{p_{11}}}{\sqrt{1 + \left(\frac{p_{21}}{p_{11}}\right)^2}}$$

Similarly for $\theta_3, \theta_4, \dots$

Equation (16) is to enable us find the angle of rotation.

Now if we are to compute all eigenvalues of the matrix given in section 2.1, we shall first reduce A (the matrix) to tri -diagonal via Householder's method to obtain

$$A_2 = \begin{bmatrix} 6 & -\sqrt{18} & 0 & 0 \\ -\sqrt{18} & 7 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 6 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Clearly the eigenvalue in A_2 is 3.

QR method is applied to a 3×3 matrix

Hence,

$$P_0 = P = \begin{bmatrix} 6 & -\sqrt{18} & 0 \\ -\sqrt{18} & 7 & \sqrt{2} \\ 0 & \sqrt{2} & 6 \end{bmatrix}$$

Multiply P by

$$C_2 = \begin{bmatrix} cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ and then

$C_2 P$ by

$$C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix}$$

$$(-\sin\theta_2).6 + (\cos\theta_2)\left(-\sqrt{18}\right) = 0$$

From (16)

$$\cos\theta_2 = 0.81649658$$

$$\sin\theta_2 = -0.57735027.$$

With these values, we have

$$C_2 P = \begin{bmatrix} 7.34846923 & -7.50555350 & -0.81649658 \\ 0 & 3.26598632 & 1.15470054 \\ 0 & 1.41421356 & 6.00000000 \end{bmatrix}$$

$$in\ C_3, \cos\theta_3 = 0.91766294$$

$$\text{Sin}\,\theta_3 = 0.39735971$$

This gives

$$R_0 = C_3 C_2 P = \begin{bmatrix} 7.34846923 & -7.50555350 & -0.81649658 \\ 0 & 3.53902608 & 3.44378413 \\ 0 & 0 & 5.04714615 \end{bmatrix}$$

From this we compute

$$P_1 = R_0 C_2^T C_3^T = \begin{bmatrix} 10.33333333 & -2.05480467 & 0 \\ -2.05480467 & 4.03508772 & 2.00553251 \\ 0 & 2.00553251 & 4.63157895 \end{bmatrix}$$

Which is symmetric and tri-diagonal.

The off-diagonal entries in P, are large in absolute value hence we proceed to the next step.

Step 2:

The new angle are

$\theta_2 = -0.196291533$ and $\theta_3 = 0.513415589$

So we have

$$R_1 = \begin{bmatrix} 10.53565375 & -2.80232241 & -0.39114588 \\ 0 & 4.08329584 & 3.98824028 \\ 0 & 0 & 3.06832668 \end{bmatrix}$$

And from this;

$$P_2 = \begin{bmatrix} 10.87987988 & -0.79637918 & 0 \\ -0.79637918 & 5.44738664 & 1.50702500 \\ 0 & 1.50702500 & 2.67273348 \end{bmatrix}$$

The off-diagonal entries are somewhat smaller in absolute value than those of P_1 but still too large for the diagonal entries to be good approximation of the eigen values of P.

Thus we continue in like manner with further steps until we are able to list the main diagonal entries and the absolutely largest off-diagonal entry which is $\left|p_{12}^{(3)}\right| = |p_{21}^{(3)}|$ in all steps.

The purpose of applying the Householder method before the QR-factorization is a substantial reduction of cost in each QR-factorization, particularly when matrix A is large [1].

The QR factorization can also be obtained using Given rotations and the classical/modified Gram Schmidt Orthogonalization. The Householder and Given QR factorization methods are backward stable [6].

3.1 COMPUTING COLUMN K

Suppose we have completed the factorization for the first $k-1$ column, column k of equation (2) reads

$a_k = R_1 k q_1 + R_2 k q_2 + \cdots + R_{k-1} k q_{k-1} + R_k k q_k$ regardless of how we choose

$R_{1k}, \ldots, R_{k-1}, k.$

The vector $\bar{q}_k = a_k - R_{1k} q_1 - R_{2k} q_2 \ldots\ldots, -R_{k-1}, kqk - 1$ will be nonzero

: $a_1, a_2, \ldots, a_k$ are linearly independent and therefore

$a_{k \notin}\, span\{a_1, \ldots, a_{k-1} = span\{q_1, \ldots q_{k-1}\}$

q_k is $\bar{q}_k$ normalized if we choose

$R_{kk} = ||\bar{q}_k||$ and $q_k = \left(\frac{1}{R_{kk}}\right) \bar{q}_k$

Also $\bar{q}_k$ and q_k are orthogonal to

$q_1 \ldots, q_{k-1}$ if we choose

$R_{1k} = q_1^T a_k, R_{2k} = q_2^T a_k, \ldots, R_{k-1}, k = q_{k-1}^T q_k$

We design an algorithm for the above assuming we are given an $m \times n$ matrix A with linearly independent columns $a_1, \ldots, a_n$

Algorithm I

For $k = 1$ to n

$R_{ik} = q_1^T a_k$
$R_{2k} = q_2^T a_k$
$\vdots$
$R_{k-1,k} = q_{k-1}^T a_k$

$\bar{q}_k = a_k - \left(R_{1k} q_1 + R_{2k} q_2 + \cdots + R_{k-1,k} q_{k-1}\right)$

$R_{kk} = ||\bar{q}_k||$

$$q_k = \frac{1}{R_{kk}} \bar{q}_k$$

Example

Consider $A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix}$

Its QR factorization is given as

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= [q_1, q_2, q_3] \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{bmatrix} = QR$$

First column of Q and R

$$q_1^2 = a_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, R_{11} = \left|\left|\bar{q}_1\right|\right| = 2,$$

$$q_1 = \frac{1}{R_{11}} \bar{q}_1 = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

2nd column of Q and R

Compute $R_{12} = q_1^T q_2 = 4$

Compute

$$\bar{q}_2 = a_2 - R_{12}q_1 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix} - 4\begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Which is normalize to get

$$R_{22} = ||\bar{q}_2|| = 2, q_2 = \frac{1}{R_{22}}\bar{q}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

3rd column of Q and R

Compute $R_{13} = q_1^T a_3 = 2$ and

$R_{23} = q_2^T a_3 = 8$

Compute $\bar{q}_3 = a_3 - R_{13}q_1 - R_{23}q_2$

$$= \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} - 2\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} - 8\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix}$$

Normalize to get

$$R_{33} = ||\bar{q}_3|| = 4, q_3 = \frac{1}{R_{33}}\bar{q}_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Final result

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix} = [q_1, \ q_2, \ q_3] \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

Complexity in cycle k of algorithm 1

k-1 inner product with a_k: $(k-1)(2m-1)$ flops.

Computation of $\bar{q}_k: 2(k-1)m$ flops

Computing R_{kk} and q_k: 3 flops

Total for cycle k: $(4m-1)(k-1) + 3$m flops

Complexity for m x n factorization:

$$\sum_{k=1}^{n}\left((4m-1)(k-1)+3m\right) = (4m-1)\frac{n(n-1)}{2} + 3mn$$

$\cong 2mn^2$ flops.

3.2 Constructing a 3 QR step factorization

We shall demonstrate the efficiency of the QR- factorization technique by constructing a 3 QR step factorization to find approximation to eigenvalue of some symmetric matrices.

This result would be achieve by considering workable examples.

Example 2:

Construct a 3 QR- step to find approximations to the eigenvalues of

(i) $A = \begin{bmatrix} 9 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (ii) $A = = \begin{bmatrix} 7.0 & 0.5 & 0 \\ 0.5 & 3.5 & 0.1 \\ 0 & 0.1 & -1.5 \end{bmatrix}$

Obviously we are going to apply the method discussed in section 2 to be able to attain the needed result.

The result for (i) is given below

$$\begin{bmatrix} 9.1585 & 0.4409 & 0 \\ 0.4409 & 4.1452 & 0.1801 \\ 0 & 0.1801 & 0.6963 \end{bmatrix}, \begin{bmatrix} 9.1893 & 0.1983 & 0 \\ 0.1983 & 4.1236 & 0.0300 \\ 0 & 0.0300 & 0.6871 \end{bmatrix}, \begin{bmatrix} 9.1955 & 0.0888 & 0 \\ 0.0888 & 4.1177 & 0.0005 \\ 0 & 0.0050 & 0.6868 \end{bmatrix}$$

For (ii) we have

$$\begin{bmatrix} 7.0533 & 0.2463 & 0 \\ 0.2463 & 3.4483 & -0.0435 \\ 0 & -0.0435 & -1.5016 \end{bmatrix}, \begin{bmatrix} 7.0661 & 0.1200 & 0 \\ 0.1200 & 3.4358 & 0.0190 \\ 0 & 0.0190 & -1.5019 \end{bmatrix},$$

$$\begin{bmatrix} 7.0691 & 0.0583 & 0 \\ 0.0583 & 3.4329 & -0.0083 \\ 0 & -0.0083 & -1.5020 \end{bmatrix}$$

4.1 Numerical experiment.

Here we use the following MATLAB codes

```
[m,n] = size (A);
```

Q = zeros (m, n);

R = zeros (m ,n);

For k = 1: n

R(1: k-1, k) = Q(: , 1 : k-1)’ * A(:, k);

V = A(:, k) – Q(:, 1: k-1) ’ * R(1: k-1, k);

R (k ,k) =norm(v);

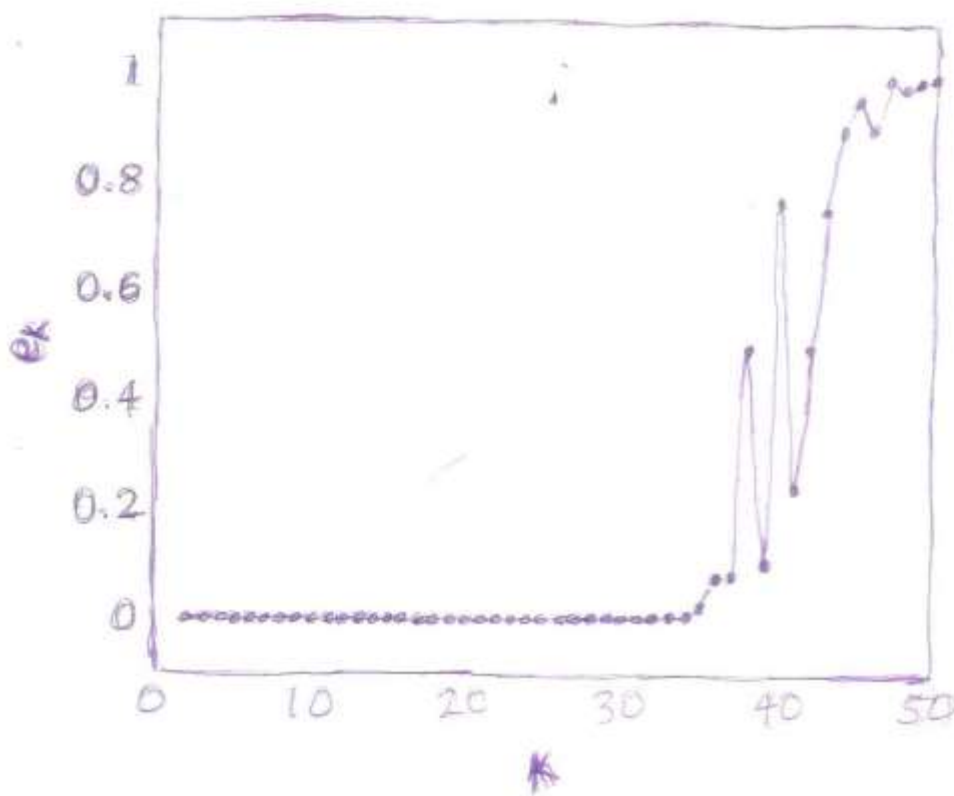
Q (: , k) = v/R(k, k);

End;

This was applied to a square matrix A of size m = n = 50. A is constructed as A = USV with U. V orthogonal, S diagonal with $S_{ii} = 10^{-10\frac{(i-1)}{(n-1)}}$, $i = 1 \ldots\ldots, n$

The graph below depicts deviation from orthogonality between q_k and previous columns.

$$e_k = \max_{0 \le i \le k} | q_i^T q_k | , \quad k = 2, \ldots \ldots, n$$



There is loss of orthogonality due to rounding errors.

4.1 Examples involving complexity

From section 3.1, complexity in cycle k, have the following dominant terms

$$(2(m-k+1)-1)(n-k+1)flops\ for\ product\ v_k^T(A_{k:m,\ k:n})$$

$$(m-k+1)(n-k+1)flops\ for\ outer\ product\ with\ v_k$$

$$(m-k+1)(n-k+1)flops\ for\ subtraction\ from\ A_{k:m,k:n}$$

Sum is roughly $4(m-k+1)(n-k+1)flops$

Therefore total for computing R and vector $v_1, \ldots\ldots, v_n$:

$$\sum_{k=1}^{n} 4(m-k+1)(n-k+1) \approx \int_0^n 4(m-t)(n-t)dt = 2mn^2 - \frac{2}{3}n^3\ flops.$$

The Householder algorithm returns the vectors $v_1, \dots\dots, v_n$ that define [Q ,$\tilde{Q}$] = $H_1\ H_2 \dots H_n$ under the Q factor. Actually there is no need to compute the matrix [Q,$\tilde{Q}$] explicitly since the vectors $v_1, \dots\dots, v_n$ are economical representation of matrix [Q ,$\tilde{Q}$].

5.1 SUMMARY AND CONCLUSION

The paper demonstrated the efficiency of the QR factorization approach by constructing a 3 QR steps to evaluate eigenvalues for general real or complex matrices. The above result was achieved by examining a real symmetric tridiagonal matrix via Householder's method. The problem of computing all eigenvalues of a real symmetric matrix was tackled by the use of QR factorization which can accurately be perform using various computer- based implementation which are applicable to every numeric algorithm, however these algorithms are cumbersome when performing manually.

Numerical experiment which illustrate the accuracy of the algorithm provided were also presented in this paper. The QR factorization approach is suitable for finding approximations for all eigenvalues of a real symmetric matrix, especially when the matrix is large.

REFERENCES

[1] G.H. Golub and C.F.Van Loan (1989).*Matrix Computations*, The Johns Hopkins University Press, Baltimore, MD. Second edition

[2] Gates, Kevin. *The QR Method for Eigenvalues.* University of Queensland, Brisbane, Australia.*http://www.maths.uq.edu.au/course/MATHS2000/linAlg/index.php?*QR Method

[3] Perry, William. QR Method. Taxas A&M University,College Station, TX *.<http://distance-ed.math.tamu.edu/Math640/chapter6/node6.html>.*

[4] Gill P.E, Golub G.H,Murray W, Saunder M.(1974). Methods for modifying matrix Factorizations.*Math.Comput.1974;28:505-535.doi:10.1090/S0025-5718-1974-0343558-6.[Cross Ref]*

[5] Anderson E, Bai Z, Dongarra J. (1992). Generalized QR factorization and its applications. *Linear Algebra Appl.1992;162:243-271.doi:10.1016/0024-3795(92)90379-0[Cross Ref]*

[6] Daniel J.W, Gragg W.B, Kaufman L, Stewart G.W.(1976). Reorthogonalization and stable algorithms for updating the Gram-Schmidt QR factorization. *Math. Comput. 1976;30:772-795.*