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On Enumeration and Characterization of Idempotent Elements in Finite Semigroup of Full Order -Preserving Contractions

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ABSTRACT
In this work, we considered the semigroup OCT_n consisting of all mappings of a finite set $X_n = \{1, 2, 3, - - - , n\}$ which are both order – preserving and contraction, that is mapping $\alpha : Xn \to Xn$ such that, for all $,y \in X_n,\ x \le y \Rightarrow x\,\alpha \le y\,\alpha, and\ |\,x\,\alpha - y\,\alpha\,| \le |\,x - y|$. In particular, we enumerates and characterizes the idempotent elements (element satisfying $\alpha^2 = \alpha$) of order preserving full contraction denoted by OCT_n

Keywords: Full Transformation, semigroup, subsemigroup, Contraction, Idempotent

1. INTRODUCTION

Semigroup theory arose as a special direction of algebra with its own objectives, formulations of problems and methods of investigating such problems. One of the main reason and motivating factor for the existence of many mathematical theories are natural examples. As the usual composition in semigroup is associative, each set of transformation closed with respect to the composition (associativity) form a semigroup called transformation semigroup. Transformation semigroups are the trivial applicant for such example in semigroup theory. Various transformation of different sets are seen everywhere in mathematics all the times. Along with this, the combinatorial aspect of semigroup theory makes a helpful contribution to the study of semigroups. This is because they give quality comprehension of the semigroup under consideration. It therefore important to study this aspect as an integral part of semigroup theory in modern times. In recent years, a lot of combinatorial results have been obtained among all the transformation semigroup, one can distinguish three classical series of semigroups. That is , the Full transformation denoted by T_n of all transformation on $X_n = \{1,2,3,\ldots,n\}$, the partial transformation P_n of all partial transformations of the set X_n and the symmetric inverse semigroup I_n of all partial one-to-one transformation of X_n. One of the main evidences for the importance of these semigroup is their universal property; every finite semigroup is realizable as a subsemigroup of some T_n; also every finite inverse semigroup is realizable as a subsemigroup of I_n

These semigroups along with many of their interesting subsemigroups have been studies both algebraically and combintorially by many authors. These studies were pioneered by Howie (1966) in which he showed that a singular elements (non – invertible elements) in Tn are generated by singular idempotents in T_n (That is singular elements $\alpha \in T_n$ satisfying $\alpha^2 = \alpha$)). Howie (1966) work drew the attention of many researchers for example Garba (1990, 1994a, b, c, d, e) (Ayik et al 2005, 2008), Umar (1992, 1993, 1994, 1996) and the reference there in. Combinatorial result pertaining to order of semigroups have been studied in the semigroups T_n and many of its notable subsemigroups. Adeshola (2012) studied some combinatorial identities in the semigroup OCT_n of all order- preserving full contractions. Also, Kabiru, Umar, Shamsuddeen and Amau (2021) studies a closed form formula for counting the number of idempotent element in finite semigroup of full transformation. They do not consider the characterization of idempotent element of order- preserving contraction through analyzing the elements of order preserving contraction. It was proved by Howie(1966) that in every finite full transformation semigroup T_n, the subsemigroup $sing_n$, of all singular self-map of T_n, is generated by its set E=E($sing_n$), of all idempotents (that is, each element of $sing_n$ is expressible as a product of idempotents in E). the analogue of this result for semigroup of singular matrices was obtained by Erdos (1967). Different kind of combinatorial problems arises from the work of Howie. Many researchers became interested in addressing these problems with respect to different kind of generating sets.

Let X_n = {1, 2, - - - - , n}. then it is not difficult to see that for the semigroup T_n , P_n , I_n we have the following orders, which may be found in (Ganyushkin and Mazorchuk (2009):

$$/T_n/= n^n$$
$$/P_n/= (n+1)^n$$
$$/I_n/=\sum_{r=0}^{n}\binom{n}{r}^2 r!$$

The number of idempotent element in the semigroup T_n is computed by Harris and Schoenfield (1967) as $/E(T_n)/= \sum_{K=1}^{n}\binom{n}{K} \; k^{n-k}$, and for P_n , I_n were obtained by Ganyushkin and Mazorchuk (2009) as $/E(P_n)/= \sum_{K=0}^{n}\binom{n}{K} \; (k+1)^{n-k}$

$$/E(I_n)/= 2^n$$

We adopt as in the literature, the notation T_n, P_n, I_n to be the semigroup of full, partial and partial one to one transformation of X_n. Algebraic and combinatorial properties of these semigroup were extensively studied in the literature, for example see Preston, Mazorchuk and Howie (Clifford and Preston, 1961; Ganyushkin and Mazorchuk, 2009; Howie,1995). For proper understanding on the concept of semigroup, we refer the intended reader to any of see Preston, Mazorchuk and Howie (Clifford and Preston, 1961; Ganyushkin and Mazorchuk, 2009; Howie,1995) as well.

Let $CT_n = \{\alpha \in T_n : /x\alpha - y\alpha/ \leq /x-y/ \; \forall \, x, y \in X_n\}$, then CT_n is known to be subsemigroup of T_n. The study of this semigroup and it subsemigroups was first initiated in 2012 by Umar and Alkharousi (2012) and supported by a grant from the Research Council of Oman (TRC). In the proposal by Umar and Alkharousi (2012), notations for the semigroups and it subsemigroups were given. In this paper we enumerates, study and come up with the general characterization of idempotent element

2. PRELIMINARIES

All definitions supplied in this section are from Howie (1995) and Ganyushkin and Mazorchuk (2009)

2.1 Semigroup: A semigroup is a set S together with an associative binary operation usually denoted by just a position, that is $(xy)x = x(yz)\ \forall x, y\ x\ \in S.$

2.2 Subsemigroup: A non-empty subset T of a semigroup S is called a subsemigroup of S, if it is closed under operation of S i.e for all $x, y\ \in\ T, xy\ \in\ T$ or T^2 is a subset of T

2.3 Full Transformation: Let $X_n = \{1,2,\dots,n\}$. A mapping α such that $dom(\alpha)$ is a subset of $X_n \rightarrow im(\alpha)$ is a subset of X_n is called full transformation of X_n if $dom(\alpha) = X_n$. The set of all full transformation of X_n, forms a semigroup under composition of mappings called full transformation semigroup.

2.4 Partial Transformation: Let $X_n = \{1,2,\dots,n\}$. A mapping α such that $dom(\alpha)$ is a subset of $X_n \rightarrow im(\alpha)$ is a subset of X_n is called partial transformation of X_n . The set of all partial transformations of X_n is denoted by P_n

2.5 Regular semigroup: An element 'a' of a semigroup S is called regular if there exists $x\ \in S$ such that $xax = a$. The semigroup S is called regular if all its elements are regular. That is, if $(\forall a\ \in S)(\exists x), axa = a$

2.6 Order- preserving mapping (OT_n): A full (partial) mapping α is said to be order-preserving if $\forall x, y\ \in\ dom(\alpha)\ x \leq y \rightarrow x\alpha \leq y\alpha$. Thus, the full order-preserving mapping is a subsemigroup of T_n denated by $O_n = \{\alpha\ \in Sing_n{:}\ (\forall x, y\ \in X_n\ \ x \leq y \rightarrow x\alpha \leq y\alpha\}$

2.7 Order- preserving contraction mapping (OCT_n): A full transformation α of a finite set X_n = $\{1,2,3,\dots,n\}$ is said to be order – preserving and contraction, if it satisfy the mapping $\alpha : Xn \rightarrow Xn$ such that, for all $, y \in X_n,\ x \leq y \Rightarrow x\alpha \leq y\alpha$, and $|x\alpha - y\alpha| \leq |x - y|$

2.8 Idempotent: An element $\alpha\ \in S$ is called an idempotent if $\alpha^2 = \alpha$. We write $E(S)$ to denote the set of all idempotent in S

Theorem 2.1 (Howie (1995)) A semigroup S has at most one identity.

Proof. If 1 and 1^1 are elements of S with property that $x1 = 1x = x\ and\ x\ 1^1 = 1^1x = x$ for all x in S, then $1^1 = 11^1$ (since 1 is an identity)

= 1 (since 1^1 is identity)

If S is a semigroup, which has no identity element, then it is very easy to adjoin an extra element 1 to S (to form a monoid out of S) given that 1s = s1 = s for all $S\ \in\ S$, and 11 = 1, it is then easy to see that $S\ \cup \{1\}$ becomes a monoid. Given monid, denoted by S^1, is defined by

$$S^1 = \begin{cases} S & if\ S\ has\ identity \\ S\ U\ \{1\} & otherwise \end{cases}$$

and called a semigroup with identity adjoined if necessary.

If a semigroup S with at least two elements contains an element O given that, for all $x\ \in S,\ 0\ x = x\ 0 = x = 0$, then s is called semigroup with zero and the element 0 as the zero element of S.

By analogy with case of S^1, for any semigroup S, we defined

$$S^0 = \begin{cases} S & if\ S\ has\ zero \\ S\ U\ \{0\} & otherwise \end{cases}$$

and refers to S^o as the semigroup obtained from S by adjoining a zero if necessary.

2.9 Subsemigroup and Ideals

A non – empty subset T of a semigroup S is called a subsemigroup of S if it is closed with respect to multiplication that is, if for all $x, y \; in \; T, xy \in T$.

A subsemigroup of S which is a group with respect to the multiplication inherited from S is called a subgroup of S.

2.10 Ideal and Green's relations

The notion of ideals lead naturally to the consideration of certain equivalence relation on a semigroup. These equivalence relations, first introduced by Green (1951) played a fundamental role in the development of semigroup theory. Since their introduction, they have become standard tools for investigating the structure of semigroups. If a is an element in a semigroup S, the sets

$S^1a = Sa \cup \{a\}$, $aS^1 = aS \cup \{a\}$ and $S^1aS^1 = SaS \cup Sa \cup a\,S \cup \{a\}$, are left, right and two – sided ideals of S respectively. These are respectively the smallest left, right and two sided. Ideals of S containing a. We shall call them principal left, right and two-sided ideals of S generated by a respectively.

For any two elements $a, b \in S$, we define the equivalences $\mathcal{L}, R, \mathcal{J}, \mathcal{H} \; and \; \mathcal{D} \; on \; \mathcal{S}$ by

$a \, \mathcal{L} \, b$ if and only if $S^1a = S^1b$

$a \, \mathcal{R} \, b$ if and only if $aS^1 = bS^1$

$a \, \mathcal{J} \, b$ if and only if $S^1aS^1 = S^1bS^1$

$H = \mathcal{L} \; n \; \mathcal{R} \quad and \quad \mathcal{D} = \mathcal{L} \, o \, \mathcal{R}$

These five equivalences are known as Green's relation (Howie, 1995).

Propositions 2.11 (Howie (1995)) let

1. $a \, \mathcal{L} \, \beta$ if and only if $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$
2. $a \, \mathcal{R} \, \beta$ if and only if $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$
3. $a \, \mathcal{J} \, \beta$ if and only if $|\, im(\alpha)| = |\, im(\beta)|$
4. $\mathcal{D} = \mathcal{J}$

3. RESEARCH METHODS

3.1 Number of order – preserving full contractions

This section is dedicated to finding an alternative method of listening/enumerating the number of elements of order – preserving full contractions. From the enumeration, we studied the idempotent element and characterized them. As a consequence of proposition 2.5, we see that, the J-classes in T_n are J_r and the number of L – classes is the number of distinct subset of X_n of cardinality r, that is, the binomial coefficient $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

The number of R-classes is the number of equivalences on X_n having r classes, that is, the stirling number of the second kind $S(n, r)$ defined recursively as $S(n, r) = S\,(n-1, r-1) + rS(n-1, r)$with boundary conditions $S(n, 1) = S(n, n) = 1$, Also, $S(n, n-1) = \frac{n\,(n-1)}{2}$ and $S\,(n, 2) = 2^{n-1}$

Therefore, a J-class J_r of T_n is visualized as an egg box in which the α - classes are the columns, the R – classes are the rows and the H - classes are the cells. The number of cells is $\binom{n}{r}$ x S (n, r), and each cell contains r! elements.

Since the semigroup OCT_n is a subsemigroup of OT_n. We obtain the elements of OCT_n for small values of n = 1, 2, 3, 4 by only considering order – preserving contraction mappings.

4. RESULT AND DISCUSSIONS IN OCT_n

For n = 1

Table 1: Elements of height 1 in OCT_1

$J_1(OCT_1)$	{1}
1	$\begin{pmatrix}1\\1\end{pmatrix}$

$$|OCT_1| = |J_1(OCT_1)| = 1$$

For n = 2

Table 2: Elements of height 1 in OCT_2

$J_1(OCT_2)$	{1}	{2}
1 2	$\begin{pmatrix}1 & 2\\ & 1\end{pmatrix}$	$\begin{pmatrix}1 & 2\\ & 2\end{pmatrix}$

Table 3: Elements of height 2 in OCT_2

$J_2(OCT_2)$	{1, 2}
1 / 2	$\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}$

$$|OCT_2| = |J_1(OCT_2| + |J_2(OCT_2)| = 2 + 1 = 3$$

For n = 3

Table 4: Elements of height 1 in OCT_3

$J_1(OCT_3)$	{1}	{2}	{3}
1 2 3	$\begin{pmatrix}1\,2\,3\\1\end{pmatrix}$	$\begin{pmatrix}1\,2\,3\\2\end{pmatrix}$	$\begin{pmatrix}1\,2\,3\\3\end{pmatrix}$

Table 5: Elements of height 2 in OCT_3

$J_2(OCT_3)$	{1,2}	{1, 3}	{2, 3}
1 / 2 3	$\begin{pmatrix}1 & 23\\1 & 2\end{pmatrix}$	$H-\text{Class}$	$\begin{pmatrix}1 & 23\\2 & 3\end{pmatrix}$
12 / 3	$\begin{pmatrix}12 & 3\\1 & 2\end{pmatrix}$	$H-\text{Class}$	$\begin{pmatrix}12 & 3\\2 & 3\end{pmatrix}$

The empty cells in the table are those H – classes of OT_n that contain no contraction mappings. This is also the case for all subsequent tables of the elements of OCT_n

Table 6: Elements of height 3 in OCT_3

J_3 (OCT_3)	{1, 2, 3}
1/2/3	$\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}$

$$|OCT_3| = |J_1(OCT_3)| + |J_2(OCT_3)| + |J_3(OCT_3)|$$
$$= 3 + 4 + 1 = 8$$

For n = 4

Table 7: Elements of height 1 in OCT_4

$J_1(OCT_4)$	{1}	{2}	{3}	{4}
1 2 3 4	$\begin{pmatrix}1\,2\,3\,4\\1\end{pmatrix}$	$\begin{pmatrix}1\,2\,3\,4\\2\end{pmatrix}$	$\begin{pmatrix}1\,2\,3\,4\\3\end{pmatrix}$	$\begin{pmatrix}1\,2\,3\,4\\4\end{pmatrix}$

Table 8: Elements of height 2 in OCT_4

$J_2(OCT_4)$	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}	{3,4}
1/ 2 3 4	$\begin{pmatrix}1 & 234\\1 & 2\end{pmatrix}$	H – Class	H – Class	$\begin{pmatrix}1 & 234\\2 & 3\end{pmatrix}$	H – Class	$\begin{pmatrix}1 & 234\\3 & 4\end{pmatrix}$
12 / 34	$\begin{pmatrix}12 & 34\\1 & 2\end{pmatrix}$	H – Class	H – Class	$\begin{pmatrix}12 & 34\\2 & 3\end{pmatrix}$	H – Class	$\begin{pmatrix}12 & 34\\3 & 4\end{pmatrix}$
123 / 4	$\begin{pmatrix}123 & 4\\1 & 2\end{pmatrix}$	H – Class	H – Class	$\begin{pmatrix}123 & 4\\2 & 3\end{pmatrix}$	H – Class	$\begin{pmatrix}123 & 4\\3 & 4\end{pmatrix}$

Table 9: Elements of height 3 in OCT_4

$J_3(OCT_4)$	{1, 2, 3}	{1, 2, 4}	{1, 3, 4}	{2, 3, 4}
1/ 2 /3 4	$\begin{pmatrix}1 & 2 & 34\\1 & 2 & 3\end{pmatrix}$	H – Class	H – Class	$\begin{pmatrix}1 & 2 & 34\\2 & 3 & 4\end{pmatrix}$
1/23 /4	$\begin{pmatrix}1 & 23 & 4\\1 & 2 & 3\end{pmatrix}$	H – Class	H – Class	$\begin{pmatrix}1 & 23 & 4\\2 & 3 & 4\end{pmatrix}$
12/ 3 / 4	$\begin{pmatrix}12 & 3 & 4\\1 & 2 & 3\end{pmatrix}$	H – Class	H – Class	$\begin{pmatrix}12 & 3 & 4\\2 & 3 & 4\end{pmatrix}$

Table 10: Elements of height 4 in OCT_4

$J_4(OCT_4)$	{1, 2, 3, 4}
1 / 2 / 3 / 4	$\begin{pmatrix}1 & 2 & 3 & 4\\1 & 2 & 3 & 4\end{pmatrix}$

$$|OCT_4| = |J_1(OCT_4)| + |J_2(OCT_4| + |J_3(OCT_4)| + |J_4(OCT_4)| = 4 + 9 + 6 + 1 = 20$$

From the last tables, we developed the following sequence of enumerations of OCT_n for small values of n and this happens for all value of $n \geq 1$. We should also note that all the element with red colour are idempotent

n	1	2	3	4
$E(OCT_n)$	1	3	6	10

The above table summarized that in full transformation semigroup, one may have element that are contractions or otherwise, these that are idempotent, but our concern is to characterise these contraction element that are idempotent. So, from the table above, T_1 indicates that we have only one element and it is a contraction and at the same time an idempotent. In T_2 indicates that we have three elements that are contraction and idempotent, while in T_3 indicates that we have eight elements that are contractions and idempotent. The target here, is to characterize the idempotent element.

Theorem 4.1 An element $\alpha \in OCT_n$ of height r is an idempotent if and only if α is of the form

$$\alpha = \begin{pmatrix} \{1,..,m-1\} & m & m+1 \ldots m+r-1 & \{m+r,\ldots,n\} \\ m-1 & m & m+1 \ldots m+r-1 & m+r \end{pmatrix}$$

Proof. It is clear that if α is of the given form, then it is an idempotent. Conversely, if $\alpha \in OCT_n$ is not of the given form, then there is at least one block of α, say A_j, with $1 < j < r$ such that A_j is not singleton. But then $im\alpha$ is not convex and so α is not an idempotent.

Now, looking at the element with red colour, that is idempotent element (satisfying $\alpha^2 = \alpha$). It can be seen that they are of the above form. While the rest of the element (element that does not have red colour) are just order preserving contraction

5. CONCLUSION AND RECOMMENDATIONS

5.1 CONCLUSION

We have shown that the semigroup OCT_n can be deducted from the subsemigroup of T_n . so, we obtain the elements of OCT_n for small value of $n = 1,2,3,4$ by extracting contraction mappings from the order – preserving mappings . The idempotent element are highlighted and their characterizations were described

5.2 RECOMMENDATIONS

Due to the importance of combinatorics which arises often from the study of finite transformation semigroups in science and technology, there is need to solve these combinatorial problems as such we recommend that similar study to be extended to each of the following transformation semigroups:

(1) The semigroup OCI_n consisting of all partial one-to-one order-preserving contraction mappings of X_n
(2) The semigroup OCP_n consisting of all partial order-preseving contraction mappings of X_n
(3) The semigroup CT_n consisting of all full contraction mappings of X_n
(4) The semigroup CP_n consisting of all partial one-one contraction mappings of X_n

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