International Journal of Innovative Mathematics, Statistics & Energy Policies 5(2):1-14, April-June, 2017

© SEAHI PUBLICATIONS, 2017 www.seahipaj.org ISSN: 2467-852X

On the Representations of Finite 2 – Groups with Minimum Centre

Jelten, B. Naphtali[1*], Stephen Y. Kutchin[2], Gusnung I. Ezekiel[3]

[1]**Department of Mathematics University of Jos, P.M.B. 2084, Plateau State, Nigeria**
[2]**Department of Mathematics University of Jos, P.M.B. 2084, Plateau State, Nigeria**
[3]**Department of Remedial Science University of Jos**
***E-mail of the corresponding author: jeltenn2012@gmail.com**

ABSTRACT
Jelten B. Naphtali has achieved the minimum number of irreducible representations of finite non commutative groups using the centre. In this paper we determine the irreducible representations of finite 2 – groups with the centre fixed at its minimum. We consider this for finite commutative and finite non commutative groups both of prime power order. That is groups of order 2^n for $1 \le n \le 6$, where n is an integer. The major instrument here is the degree equation. The scheme in this paper is simplified as it is reduced to a single variable. This makes it easier to manipulate and apply. The same results is obtained with this scheme as that of Jelten and Apine.
Keywords: centre, commutative, p – group, centralizer. conjugacy

1. INTRODUCTION
Here we define some basic concepts, state some fundamental theorems and prove the basic ones that we need in our results.

1.1 Definition

A group is a non-empty set G on which is defined a rule for combining elements $a, b \in G$ such that the following axioms are satisfied for all $a, b \in G$:

1. $ab \in G$
2. $a(bc) = (ab)c$
3. there exists an element $e \in G$ called the identity element in G such that: $ae = a = ea$
4. there exists an element $a^{-1} \in G$ called the inverse element a of G such that:

   $aa^{-1} = e = a^{-1}a$.

A set G which satisfies axiom 1 only is called a groupoid. If a groupoid satisfies axiom 2 it is called a semi group. A semi group which satisfies axiom 3 is said to be a monoid.

While the commutative law may not hold throughout the group it nevertheless holds for some pairs of elements in G, hence we have the next definition.

1.2 Definition

A group G with the property that $ab = ba$ for every pair of elements $a,b \in G$ is said to be a commutative group. A group in which there exists a pair of elements a, $b \in G$ endowed with the property that $ab \neq ba$ is called a non non commutative group. Consequently the property which a, b of G are endowed with when they are combined, determines the distinguishing properties of such a group.

We have that a group consists of smaller groups and we give an analogous definition of subsets in groups in the next definition.

1.3 Definition

A non empty subset N of a group G is said to be a subgroup of G written $N \leq G$, if N is a group under the operation inherited from G. If $N \neq G$, then N is a proper subgroup of G. We call G a simple group if the only normal subgroups of G are the identity subgroup and the subgroup G itself.

A subgroup N of G such that every left coset is a right coset and vice versa is called a normal subgroup of G. That is $Nx = xN$ or $x^{-1}Nx \leq N$ and we write $N \triangleleft G$. If G is commutative then every subgroup of G is normal.

1.4 Definition

The number of elements in a group G written $|G|$ is called the cardinality of the group. If G is finite of order n we write $|G| = n$ otherwise, $|G| = \infty$ if G has infinite order.

The least number n if it exists such that $a^n = 1$for a in G is called the order of a and we write $o(a) = n$ and $o(a) = \infty$ if no such n exists. An element of order two is said to be an involution that is $o(a) = 2$ then $a = a^{-1}$ so a and a^{-1} have the same order.

The identity element is the only element of order 1 However, if a, $b \in G$ are such that $a = g^{-1}bg$ for some g in G then a and b are conjugate elements with $o(a) = o(b)$. Furthermore the order of an element of G divides $|G|$. In particular $a^{|G|} = 1$ where G is finite.

1.5 Definition

A group G is said to be cyclic if it is generated by a single element say a and we write $G = \langle a \rangle$. That is $G = \{a^n : n \in Z\}$ in the finite case. In the infinite case, G is cyclic if the powers of $a \in G$

exhaust G. That is, every element $b \in G$ can be expressed as $b = a^m$ for some integer m. In this sense we say $a \in G$ generates G. An important fact about cyclic groups is that they are commutative.

Every group of prime order is cyclic. Equivalently every subgroup of a cyclic group is cyclic.

A corollary of this is:

1.6 Corollary

(i) If G is a group such that $g^2 = 1$ for all g in G then G is commutative.

(ii) $g^n = 1$ if and only if $(x^{-1}gx)^n = 1$

1.7 Definition

Let G be a group and $H < G$. For $q \in G$ the subset $Hq = \{hq{:}h \in H\}$ of G is called a right coset of H in G. Distinct right cosets of H in G form a partition of G. That is, every element of G is precisely in one of them. Left coset is similarly defined. If G is commutative we talk of cosets of H. The number of distinct right cosets of H in G is written as $|G{:}H|$ and is called the index of H in G. If G is finite so is H and G is partitioned into $|G{:}H|$ cosets each of order $|H|$. It follows that: $|G{:}H| = \frac{|G|}{|H|}$.

We note from this definition that $|H|$ and $|G{:}H|$ divide $|G|$. A consequence of this is that if H is a subgroup of G then, $|G| = |G{:}H||H|$.

This naturally leads to an important theorem in group theory: The Lagrange's Theorem which is next.

1.8 Theorem

If a group G is finite and H is a subgroup of G then the order of H divides the order of G.

Proof

By definition 1.7 we have that the right cosets of H form a partition of G. Thus each element of G belongs to at least one right coset of H in G and no element can belong to two distinct right cosets of H in G. Therefore every element of G belongs to exactly one right coset of H. Moreover each right coset of H in G contains $|H|$ elements. Therefore if the number of right cosets of H in G is n, then $|G| = n|H|$. Hence the order of H divides the order of G.

1.9 Remark

The converse of Lagrange's Theorem is not true in general. That is, a group need not have a subgroup of order *m* if *m* is a divisor of the order of *G*. A counter example is a group of order 12 which has no subgroup of order 6, as in the case of the alternating group of degree 4.

Theorem 1.8 has the following corollary.

1.10 Corollary

If *G* is a finite group and *q* is in *G* then o(*q*) is a divisor of the order of *G* since <q> is a subgroup of G generated by q.

Next we define an important concept.

1.11 Definition

Let a, q $\in$G. Then a is conjugate to q in G if there exist an element g$\in$G such that $q=g^{-1}ag$. The set of all elements of G that are conjugate to a in G is called the conjugacy class of a in G which we denote by C(a). And as such: $C(a) =\{g^{-1} ag:g\in G\}$

We note that C(a) is a subgroup of G and by theorem 1.8 its order divides that of G. Subgroups belonging to the same conjugacy class are conjugates. However conjugate elements lie in the same conjugacy class and have the same order.

1.12 Definition

The centre Z(G) of a group G is the set of all elements z in G that commute with every element q in G. We write $Z(G)=\{z\in G:zq=qz, \text{ for all } q\in G\}$

1.13 Remark

Z(G) is a commutative normal subgroup of G. For commutative and non commutative groups the minimum size of the centre is 1and 2 respectively. The centre of a group G is its subgroup of largest order that commute with every element in the group.

1.14 Definition

The centralizer $C_G(q)$ of an element q in G is the set of all elements g $\in$G that commute with q. That is: $C_G(q) =\{g\in G:gq=qg, \text{ for some } q\in G\}$.

1.15 Remark

The centralizer is a subgroup of G but not a normal subgroup in general and hence its order divides |G|. The index of $C_G(q)$ in G is the size of the conjugacy class C(q) of q in G. That is

$|C(q)|=|G:C_G(q)|$. If $q \in Z(G)$ then $|C(q)|=1$ and $q^{-1}gq=q$. So that $C_G(q) = G$. Consequently the quotient of G by $C_G(q)$ is not a group The identity element belongs to its own class. If the group is commutative then, $C(a)=\{a\}$ for all *a* in *G*. This shows that the concept of conjugacy is trivial in the commutative case.

Generally $|C| = |G:C_G(a)|$. Accordingly the size of the conjugacy class divides the order of *G*. If we choose a single representative element x_i from each conjugacy class, then by the fact that the conjugacy classes are disjoint we have that: $|G| = \sum|G:C_G(x_i)|$. Its is known that the centre of a group is the union of conjugacy classes of size one. This gives rise to an important theorem. The class equation: $|G| = |Z(G)|+ \sum|G:C_G(x_i)|$

1.16 Theorem

All elements of a conjugacy class have the same order. The converse of this is false.

1.17 Remark

Elements of the same order in a group are not always conjugates. This is obvious in the commutative case where different elements are not conjugates.

1.18 Definition

A *p* - group is a group whose order is a power of a prime *p*. If a group *G* has order p^m where *p* is a prime and *m* is a positive integer, then *G* is a *p* - group. Consequently 2- groups are p – groups with p = 2.

We collect properties of *p* - groups in the following.

1.19 Lemma

Let *G* be a group of order p^n, with $n \geq 1$ then:

1. If $\{1\} \neq H \lhd G$, then $H \cap Z(G) \neq \{1\}$. In particular $Z(G) \neq \{1\}$;
2. If $K \leq Z(G)$ and *G/K* is cyclic then *G* is commutative;

3. If $n \leq 2$, then G is commutative

1.20 Definition

The subgroup $G^{/}$ or $[G,G]$ of a group G generated by the elements of the form $sts^{-1}t^{-1}$, for all $s, t \in G$ is called the derived group or commutator subgroup of G. We write $[s,t] = sts^{-1}t^{-1}$ and call this the commutator of s and t. Thus: $G^{/} = \{ [s,t] : s, t \in G \}$.

The derived $G^{/}$ is normal in G. It is the largest abelian quotient of G.

1.21 Theorem

The derived subgroup $G^{/}$ of a finite group G is the unique minimal normal subgroup of G such that $G/G^{/}$ is commutative. That is G/N is commutative implies that $G^{/} \leq N$ and $G/G^{/}$ is commutative.

Proof

$G^{/}$ is normal in G. Let N be normal in G and $a, q \in G$. Then $aqa^{-1}q^{-1} \in N$ imply that $aqN = qaN$. This also imply that $(aN)(qN) = (qN)(aN)$. So $G^{/} \leq N$ imply G/N is commutative. Since $G^{/}$ is normal in G, $G/G^{/}$ is a commutative group.

The next Theorem stated here for reference is the degree equation from [7].

1.22 Theorem

Let G be a group and the degrees of inequivalent irreducible representations of G be r_i for $1 \leq i \leq |C|$, then:

(i) $|G| = \sum r_i^{\,2}$;

(ii) $|C|$ is the number of conjugacy classes of G;

(iii) $|G/G^{/}|$ is the number of $r_i = 1$;

(iv) Each r_i divides $|G/Z(G)|$.

We note from Definition 1.20 and Theorem 2.5.9 that the degree equation becomes.

1.23 Theorem

$$|G| = |G{:}G^{/}| + \sum r_i^{\,2}, \quad |G{:}G^{/}|+1 \leq i \leq |C|.$$

The next corollary outlines properties of conjugacy class proved by [3]

1.24 Corollary

If G is a finite group, then:

(i) every group is a union of its conjugacy classes and distinct conjugacy classes are disjoint;

(ii) conjugacy class is an equivalence relation where the equivalence classes are the conjugacy classes.

A relationship between the centre of G and the centralizer of the elements of G is given by:

1.25 Lemma

The centre Z(G) of a group G is the intersection of the centralizers $C_G(a)$ of elements a in G.

From [2] we have a theorem on the cardinality of the conjugacy class of an element of *G*.

1.26 Theorem

Let G be a finite group and $q \in G$, then the conjugacy class C(q) of q in G is given by:

$|C(q)| = |G{:}C_G(q)| = \frac{|G|}{|C_G(q)|}$. The sum of the centralizers of all elements of *G* can be separated into sums of the centralizers of all the elements from each conjugacy class of *G*.

Next we relate normal subgroup to conjugacy class as:

1.27 Proposition

Let $N \leq G$, then $N \lhd G$ if and only if *N* is the union of conjugacy classes of *G*.

Proof

If *N* is the union of the conjugacy classes of *G*, then for $n \in N$, $q \in G$ we have

$q^{-1}nq \in N$. So $q^{-1}Nq \leq N$. Conversely if $N \lhd G$ then for all $n \in N$, $q \in G$ we have that $q^{-1}nq \in N$. This implies that $C(n) \leq G$ and so $N = \bigcup_{n \in N} C(n)$. Hence the result.

[6] proves the following theorem.

1.28 Theorem

If a finite group *G* has a centre *Z*(*G*) and *G/Z*(*G*) is cyclic then *G* is commutative.

A statement of the consequence of Theorem 1.8 is:

1.29 Corollary

The order of an element a in G divides the order of G since $<a>$ is a subgroup of G generated by a.

1.30 Proposition

If the order of a finite group G is a power of a prime p then G has a non trivial centre. Equivalently the centre of a p - group contain more than one element.

Proof

Let G be the union between its centre and the conjugacy classes say J_i of size greater than 1.

Then from remark 1.15 $|G| = |Z(G)|+\sum|C(J_i)|$

Each conjugacy class J_i has size of a power w say of prime p such that $w \geq 1$. In this case $w = 0$ for the conjugacy classes whose elements are central elements. Since each conjugacy class J_i has size a power of p then $|J_i|$ is divisible by p. Furthermore as p divides $|G|$, it follows that p also divides $|Z(G)|$. Accordingly $Z(G)$ is non- trivial.

Proposition 1.30 implies that there are elements of G other than the identity that commute with every element of G.

1.31 Theorem

Let G be a finite group the $|G| = \sum|G: C_G(q_i)|$ .... (i),

where the sum runs over the elements from each conjugacy class of G.

We note that from Theorem 1.26, equation (i) becomes

$$|G| = |Z(G)| + \sum|G: C_G(q_i)| \quad \text{...} \quad (ii)$$

Here the sum in (ii) runs over q_i from each conjugacy class such that q_i is not an element of $Z(G)$. From theorem 1.26 and equation (ii) above we have:

$$|G| = |Z(G)| + \sum|C(q_i)| \quad \text{..} \quad (iii)$$

1.32 Remark

In the commutative environment, the sum in equation (iii) of Theorem 1.31 is zero. Consequently, the class equation is relevant only when we are in the non commutative environment. The fact that each element of $Z(G)$ forms a conjugacy class containing just itself gives rise to the class equation. From Theorems 1.8 and 1.31 we have:

1.33 Proposition

If the order of a finite group G is a power of a prime p then G has a non trivial centre. Equivalently the centre of a p - group contain more than one element.

Proof

Let G be the union between its centre and the conjugacy classes say J_i of size greater than 1.

Then from equation (*iii*) of Theorem 1.31, $|G| = |Z(G)|+\sum|C(J_i)|$

Each conjugacy class J_i has size of a power w say of prime p such that $w \geq 1$. In this case $w = 0$ for the conjugacy classes whose elements are central elements. Since each conjugacy class J_i has size a power of p then $|J_i|$ is divisible by p. Furthermore as p divides $|G|$, it follows that p also divides $|Z(G)|$. Accordingly $Z(G)$ is non- trivial.

From [3] and [1] we have:

Theorem 1.34

Every group of order p^2 where p is a prime number is commutative.

Proof

By Proposition 1.33, the centre is non trivial. Thus by Theorem 1.8 the only remaining possibilities are $|Z(G)| = p$ or p^2. If $|Z(G)| = p^2$, then we are done. If however $|Z(G)| = p$, then there would exist some a in G such that a in not in the centre of G. For a we have that $Z(G) \subseteq C_G(a)$ and $|Z(G)| \neq |C_G(a)|$because the centralizer of a contains a. That forces $C_G(a) = G$, which means that a is in the centre. This contradicts our hypothesis. The contradiction shows that the case $|Z(G)| = p$ cannot occur.

From [8] more properties of the derived group include:

1.35 Theorem

If G is a finite group then:

(i) the derived group $G^/$ is a normal subgroup of G and $G/G^/$ is commutative

(ii) if H is any normal subgroup of G such that G/H is commutative then $G^/ \leq H$.

First from [2] we have:

1.36 Theorem

If G is a finite non commutative group, then the maximum possible order of the centre of G is ¼|G|.That is, $|Z(G)| \leq 1/4|G|$.

The next lemma gives the relationship between the centralizer of an element and the size of finite group G as in [2]

1.37 Lemma

Let G be a finite non commutative group and $t \in G$ such that $t \notin Z(G)$, then:

$$C_G(t) = |G|/2.$$

In the next theorem [4] determine the minimum and maximum number of irreducible representations of prime degree of non commutative group using its centre. This requires the theorem 1.36 and lemma 1.37

1.38 Proposition

(i) The number of the irreducible representations of any group G is equal to the number of conjugacy classes of G;

(ii) Every irreducible representation of a commutative group G over C, the set of complex numbers is one dimensional.

1.39 Theorem

Let G be a finite non abelian group of order r^w such that $|Z(G)| = r^t$ with $t < w$, r a prime number, t and w positive integers. Then G:

(i) does not have an irreducible representation of degree r greater than 1 whenever $t = 0$;

(ii) has its minimum number of irreducible representations of degree r greater than 1 whenever $w = 3$.

Proof

(i) If $t = 0$ the centre is trivial. From proposition 1.33 it follows that G cannot have a trivial centre when $w > 1$. Accordingly G has $|G|$ conjugacy classes and $|G|$ irreducible representations each of which according to proposition 1.38 cannot be greater than 1 whenever $t = 0$.

(ii) Since G is non abelian, there exists an element $g \in G$ such that X_c(g) = r > 1 (3.6.1). Let s be the number of irreducible representations of degree r with $s = t < w$. Then from 3.1.6,

$$|Z(G)| \leq 1/4\,|G|.$$

That is $r^t \leq 1/4\, r^w$. This implies that: $4 \leq r^{w-t}$ or $2^2 \leq r^{w-t}$.

To find the minimum irreducible representation we let r take its minimum value which is 2 (minimum prime). Then we have that $t = w - 2$. It follows that: $s = t = w - 2$.

However $w \neq 0, 1, 2$ since $t > 0$ and $s > 0$. Accordingly, the minimum value of w is 3 which gives the corresponding minimum values of s and t as $t = 1$, $s = 1$ and $r > 1$.

What follows is a theorem in which [5] count the conjugacy classes stated here for reference.

1.40 Theorem

Given that a finite group G is of prime power order with centre Z(G), then we count the number of conjugacy classes from the centralizer as follows: $|C| \leq \frac{1}{4}(3|G| - |C_G(x)|)$.

$$|G| = |Z(G)| + \sum_{i=1+|Z(G)|}^{|C|-|Z(G)|} |G : C_G(x_i)|, \;\; x \notin Z(G) \;\; \text{with } |C(x)| \geq 2.$$

So that $|G| \geq |Z(G)| + 2(|C| - |Z(G)|)$

$|G| \geq \frac{1}{2}C_G(x) + 2|C| - 2|Z(G)|$. That is $|G| \geq \frac{1}{2}C_G(x) + 2|C| - \frac{1}{2}|G|$

This leads to $2|G| \geq C_G(x) + 4|C| - |G|$, so $3|G| \geq C_G(x) + 4|C|$

We have $\frac{3|G| - |C_G(x)|}{4} \geq |C|$ which reduces to $\frac{1}{4}(3|G| - |C_G(x)|) \geq |C|$

That is $|C| \leq \frac{1}{4}(3|G| - |C_G(x)|)$

2. Our Result

2.1 Theorem

Let G be a finite p-group with centre |Z(G)|. Then the number of irreducible representation of G is |G| if the group G is abelian with centre at its minimum and $G^{/} \leq G$.

Proof

From the degree equation we have that $|G| \geq |G:G^{/}| + \sum_{i=|G:G^{/}|+1}^{|C|} (x_i)^2$

As $G^{/} \leq G$, we have without loss of generality that $|G| \geq |G:Z(G)| + \sum_{i=|G:Z(G)|+1}^{|C|} (x_i)^2$

That is $|G| \geq |G:Z(G)| + 4[|C| - |G:Z(G)|]$, so $|G| \geq |G:Z(G)| + 4|C| - 4|G:Z(G)|$

From the hypothesise the inequality becomes: $|G| \geq |\frac{G}{Z(G)}| + 4|C| - 4|\frac{G}{Z(G)}|$

As G is abelian the minimum size of the centre for such groups is 1, hence we have:

$|G| \geq |\frac{G}{1}| + 4|C| - 4|\frac{G}{1}|$

$|G| \geq |G| + 4|C| - 4|G|$, which gives $|G| \geq 4|C| - 3|G|$

$4|G| \geq 4|C|$ and the result follows. That is: $|G| \geq |C|$. Consequently $|C| = |G|$

Characteristically this is the property of finite abelian groups. so this provides an entirely new method of the prove.

2.2 Theorem

Let G be a finite non abeian 2-group with centre |Z(G)| fixed at its minimum. Then the number of irreducible representation of G is given by $|C| \leq |\frac{|G|+12}{4}|$ given that $G^{/} \leq G$.

Proof

From the degree equation we have that $|G| \geq |G:G^{/}| + \sum_{i=|G:G^{/}|+1}^{|C|} (x_i)^2$, with $x \notin |Z(G)|$

As $G^{/} \leq G$, we have without loss of generality that $|G| \geq |G:Z(G)| + \sum_{i=|G:Z(G)|+1}^{|C|} (x_i)^2$

$|G| \geq |G:Z(G)| + 4[|C| - |G:Z(G)|]$

But as G is finite from hypothesis, we have that: $|G| \geq |\frac{|G|}{|Z(G)|}| + 4[|C| - \frac{|G|}{|Z(G)|}]$

for non abelian groups $|Z(G)| \geq 2$ .and by hypothesis $|Z(G)| = 2$. Hence, the last inequality becomes: $|G| \geq |\frac{|G|}{2}| + 4[|C| - \frac{|G|}{2}]$

From lemma 1.37 $\frac{|G|}{2} = |C_G(x)|$ such $x \notin |Z(G)|$ so substituting we obtain

$|G| \geq |C_G(x)| + 4[|C| - |C_G(x)|]$

$|G| \geq |C_G(x)| + 4|C| - 4|C_G(x)|$

$|G| \geq 4|C| - 3|C_G(x)|$

That is: $|G| + 3|C_G(x)| \geq 4|C|$

Now from this inequality we have $|\frac{|G| + 6|Z(G)|}{4}| \geq ||C|$

Fixing the centre at its minimum for the group from the hypothesis, we get

$\frac{|G| + 12}{4} \geq ||C|$

Discussion

We consider the result in theorem one as trivial though fundamental in the theory of finite groups and useful in the classification of finite groups. Next we illustrate theorem 2 in what follows.

Since the group under consideration is of prime power order we have that $|G| = 2^n$ with $n > 2$. The cases for $2 < n < 6$ were considered as follows:

For $n = 3$, $|G| = 8$ hence $|C| \leq \frac{8+12}{4} = 5$

For $n=4$, $|G|=16$ hence $|C| \leq \frac{16+12}{4} = 7$

For $n=5$, $|G|=32$ hence $|C| \leq \frac{8+12}{4} = 11$

These agree with results obtained by [4].

CONCLUSION

The values obtained in the discussion section agree with those of Jelten Naphtali for the non abelian case. An interesting part of this result is that with the centre fixed at its minimum, we obtained the corresponding minimum number of the irreducible representations for the 2 – groups under consideration. Our result unlike that of Jelten Naphtali is simple with reduced number of variable and so easy to apply.

REFERENCES

Houshang, B. and Hamid, M. (2009). A note on p-groups of order $\leq p^4$. Proc. Indian Acad. Sci. (Math. Sci.), **(2009),** vol 2, 137-143.

C. Cody, Commutativity in non abelian groups. www.whitman.edu/mathematics/.../2010/seniorproject.codyclifton, (2010) 1-18. Retrieved on 2/9/ 2012.

Gordon, J and Martin, L. (2001) Representation and characters of groups (second edition). Cambridge University press, Naphtali J. B. (2016). ., The irreducible components of the representations of finite non abelian groups. Unpublished Ph. D thesis, Mathematics Department, University of Jos. Nigeria,

Naphtali J. B. and E, Apine (2015) Counting the conjugacy classed of finite groups from the Centralizer. International journal of mathematics and statistics invention, vol. 3 no. 1, pages 38 – 47.

R. Mark, (2011). Notes on group theory. https://www2,bc.edu/-/development-group- Theoryhtml, (2011), p. 4-9. Retrieved on 9/9/ 2012.

S. Louis, Introduction to abstract algebra. New York: McGraw – Hill Inc, (1975)

W. Laderman and Weir, A J Introduction to group theory, (2nd edition). Addison Wesley Longman Inc. (1966). New York.